Algorithmica (2014) 70:326–339
DOI 10.1007/s00453-013-9811-2

The Feedback Arc Set Problem with Triangle Inequality Is a Vertex Cover Problem

Monaldo Mastrolilli

Received: 2 November 2012 / Accepted: 27 June 2013 / Published online: 13 July 2013

Abstract We consider the (precedence constrained) Minimum Feedback Arc Set problem with triangle inequalities on the weights, which finds important applications in problems of ranking with inconsistent information. We present a surprising structural insight showing that the problem is a special case of the minimum vertex cover in hypergraphs with edges of size at most 3.

Keywords Feedback arc set problem · Approximation algorithms · Integer linear program formulation

1 Introduction

The MINIMUM FEEDBACK ARC SET problem (MINFAS) is a fundamental and classical combinatorial optimization problem that finds application in many different settings that span from circuit design, constraint satisfaction problems, artificial intelligence, scheduling, etc. (see e.g. Chap. 4 in [23] for a survey). For this reason it has been deeply studied since the late 60's (see, e.g., [22]).

Its input consists of a set of vertices V and nonnegative weights $\{w_{(i,j)} : (i, j) \in V \times V\}$ for every oriented pair of vertices. The goal is to find a permutation π that minimizes $\sum_{\pi(i)<\pi(j)} w_{(i,j)}$, i.e. the weight of pairs of vertices that comply with the permutation. (Different, but equivalent formulations are often given for the problem: Usually the goal is defined as the minimization of the weight of pairs of vertices *out*

This research is supported by Swiss National Science Foundation project N. 200020-144491/1 "Approximation Algorithms for Machine Scheduling Through Theory and Experiments" and by Hasler Foundation Grant 11099.

M. Mastrolilli (✉)
IDSIA, USI-SUPSI, Lugano, Switzerland
e-mail: monaldo@idsia.ch

of order with respect to the permutation, i.e. $\sum_{\pi(i)<\pi(j)} w_{(j,i)}$; Clearly by swapping appropriately the weights we obtain the equivalence of the two definitions.)

A *partially ordered set* (*poset*) $\mathbf{P} = (V, P)$, consists of a set V and a partial order P on V, i.e., a reflexive, antisymmetric, and transitive binary relation P on V, which indicates that, for certain pairs of elements in the set, one of the elements precedes the other. In the *constrained* MINFAS (see [27]) we are given a partially ordered set $\mathbf{P} = (V, P)$ and we want to find a linear extension of $\mathbf{P}$ of minimum weight.

MINFAS was contained in the famous list of 21 NP-complete problems by Karp [17]. Despite intensive research for almost four decades, the approximability of this problem remains very poorly understood due to the big gap between positive and negative results. It is known to be at least as hard as vertex cover [16], but no constant approximation ratio has been found yet. The best known approximation algorithm achieves a performance ratio $O(\log n \log\log n)$ [11, 12, 25], where n is the number of vertices of the digraph. Closing this approximability gap is a well-known major open problem in the field of approximation algorithms (see e.g. [29], p. 337). Very recently and conditioned on the Unique Games Conjecture, it was shown [14] that for every constant $C > 0$, it is NP-hard to find a C-approximation to the MINFAS.

Important ordering problems can be seen as special cases of MINFAS with restrictions on the weighting function. Examples of this kind are provided by ranking problems related to the aggregation of inconsistent information, that have recently received a lot of attention [2–4, 18, 27, 28]. Several of these problems can be modeled as (constrained) MINFAS with weights satisfying either *triangle inequalities* (i.e., for any triple i, j, k, $w_{(i,j)} + w_{(j,k)} \geq w_{(i,k)}$), and/or *probability constraints* (i.e., for any pair i, j, $w_{(i,j)} + w_{(j,i)} = 1$). Ailon, Charikar and Newman [4] give the first constant-factor randomized approximation algorithm for the unconstrained MINFAS problem with weights that satisfy either triangle inequality constraints, probability constraints, or both. For the same problem Ailon [2] gives a 3/2-approximation algorithm and van Zuylen and Williamson [28] provide a 2-approximation algorithm for the constrained version. These are currently the best known results for the (constrained) MINFAS with triangle inequalities and are both based on solving optimally and rounding a "natural" linear program relaxation (see the linear program (1a)–(1e) defined in the following). When the probability constraints hold, Mathieu and Schudy [18] obtain a PTAS.

Another prominent special case of MINFAS with restrictions on the weighting function is given by a classical problem in scheduling, namely the precedence constrained single machine scheduling problem to minimize the weighted sum of completion times, denoted as $1|\text{prec}|\sum w_j C_j$ (see e.g. [21] and [15] for a 2-approximation algorithm). This problem can be seen as a constrained MINFAS where the weight of arc (i, j) is equal to the product of two numbers p_i and w_j: p_i is the processing time of job i and w_j is a weight associated to job j (see [5, 7, 9, 10, 20] for recent advances). In [5, 10], it is shown that the structure of the weights for this problem allows for all the constraints of size strictly larger than two to be ignored, therefore the scheduling problem can be seen as a special case of the vertex cover problem (in normal graphs). The established connection proved later to be very valuable both for positive and negative results: studying this graph yielded a framework that unified and improved upon previously best-known approximation algorithms [7, 20]; moreover, it helped to obtain the first inapproximability results for this old problem [7–9] by revealing more of its structure and giving a first answer to a long-standing open question [24].

New Results The (constrained) MINFAS can be described by the following natural (compact) ILP using linear ordering variables $\delta_{(i,j)}$ (see e.g. [28]): variable $\delta_{(i,j)}$ has value 1 if vertex i precedes vertex j in the corresponding permutation, and 0 otherwise.

$$
\begin{aligned}
\text{[FAS]} \qquad \min \quad & \sum_{i\neq j} \delta_{(i,j)} w_{(i,j)} && && \text{(1a)}\\
\text{s.t.} \quad & \delta_{(i,j)} + \delta_{(j,i)} = 1, && \text{for all distinct } i, j, && \text{(1b)}\\
& \delta_{(i,j)} + \delta_{(j,k)} + \delta_{(k,i)} \geq 1, && \text{for all distinct } i, j, k, && \text{(1c)}\\
& \delta_{(i,j)} = 1, && \text{for all } (i, j) \in P, && \text{(1d)}\\
& \delta_{(i,j)} \in \{0, 1\}, && \text{for all distinct } i, j. && \text{(1e)}
\end{aligned}
$$

Constraint (1b) ensures that in any feasible permutation either vertex i is before j or vice versa. The set of Constraints (1c) is used to capture the transitivity of the ordering relations (i.e., if i is ordered before j and j before k, then i is ordered before k, since otherwise by using (1b) we would have $\delta_{(j,i)} + \delta_{(i,k)} + \delta_{(k,j)} = 0$ violating (1c)). Constraints (1d) ensure that the returned permutation complies with the partial order P. The constraints in (1a)–(1e) were shown to be a minimal equation system for the linear ordering polytope in [13].

To some extent, one source of difficulty that makes the MINFAS hard to approximate within any constant is provided by the equality in Constraint (1b). To see this, consider, for the time being, the unconstrained MINFAS. The following covering relaxation obtained by relaxing Constraint (1b) behaves very differently with respect to approximation.

$$
\begin{aligned}
\min \quad & \sum_{i\neq j} \delta_{(i,j)} w_{(i,j)} && && \text{(2a)}\\
\text{s.t.} \quad & \delta_{(i,j)} + \delta_{(j,i)} \geq 1, && \text{for all distinct } i, j, && \text{(2b)}\\
& \delta_{(i,j)} + \delta_{(j,k)} + \delta_{(k,i)} \geq 1, && \text{for all distinct } i, j, k, && \text{(2c)}\\
& \delta_{(i,j)} \in \{0, 1\}, && \text{for all distinct } i, j. && \text{(2d)}
\end{aligned}
$$

Problem (2a)–(2d) is a special case of the vertex cover problem in hypergraphs with edges of sizes at most 3. It admits "easy" constant approximate solutions (i.e. a trivial primal-dual 3-approximation algorithm, but also a 2-approximation algorithm for *general weights* (no triangle inequalities restrictions) by observing that the associated vertex cover hypergraph is 2 colorable and using the results in [1, 19]); Vice versa, there are indications that problem (1a)–(1e) may not have any constant approximation [14]. An interesting question is to understand under which assumptions on the weighting function the covering relaxation (2a)–(2d) represents a "good" relaxation for MINFAS.

Surprisingly, we show that the covering relaxation (2a)–(2d) is an "optimal" relaxation, namely, a *proper* formulation, for the unconstrained MINFAS when the weights satisfy the triangle inequalities. More precisely, we show that any α-approximate solution to (2a)–(2d) can be turned in polynomial time into an α-approximate solution to (1a)–(1e), for any $\alpha \geq 1$ and when the weights satisfy the triangle inequalities. We also observe that the same result does not hold when the weights satisfy the probability constraints (see Appendix A and B).

Interestingly, a compact covering formulation can be also obtained for the more general setting with precedence constraints. In this case we need to consider the following covering relaxation which generalizes (2a)–(2d) to partially ordered sets $\mathbf{P} = (V, P)$.

$$\min \quad \sum_{i \neq j} \delta_{(i,j)} w_{(i,j)} \tag{3a}$$

$$\text{s.t.} \quad \delta_{(x_1,y_1)} + \delta_{(x_2,y_2)} \geq 1, \qquad (x_2, y_1), (x_1, y_2) \in P, \tag{3b}$$

$$\delta_{(x_1,y_1)} + \delta_{(x_2,y_2)} + \delta_{(x_3,y_3)} \geq 1, \qquad (x_2, y_1), (x_3, y_2), (x_1, y_3) \in P, \tag{3c}$$

$$\delta_{(i,j)} \in \{0, 1\}, \qquad (i, j) \in \mathrm{inc}(\mathbf{P}), \tag{3d}$$

where $\mathrm{inc}(\mathbf{P}) = \{(x, y) \in V \times V : (x, y), (y, x) \notin P\}$ is the set of *incomparable pairs* of $\mathbf{P}$. When the poset is empty, then (3a)–(3d) boils down to (2a)–(2d) (since P is a reflexive binary relation). Note that (3a)–(3d) is a relaxation to constrained MINFAS, since Constraint (3b) and (3c) are valid inequalities (otherwise we would have cycles).

Recall that a function $w : V \times V \to \mathbf{R}$ is *hemimetric* if for all i, j, k the following is satisfied:

1. $w(i, j) \geq 0$ (*non-negativity*),
2. $w(i, i) = 0$,
3. $w_{(i,k)} \leq w_{(i,j)} + w_{(j,k)}$ (*triangle inequality*).

The following theorem summarizes the main result of the paper.

Theorem 1 *If the weighting function $w : V \times V \to \mathbf{R}$ is hemimetric then any solution to* (3a)–(3d) *can be transformed in polynomial time into a feasible solution to* (1a)–(1e) *without deteriorating the objective function value.*

We emphasize that a straightforward application of Theorem 1 does not imply a better approximation algorithm for the (constrained) MINFAS with triangle inequality. However, Theorem 1 gives a new surprising structural insight that opens the road to studying the problem under a new light which can benefit from the vast literature and techniques developed for covering problems (this was actually the case for the previously cited scheduling problem [5, 7–10, 20] where the vertex cover insight was essential to obtain improved lower/upper bounds on the approximation ratio).

The arguments that we use to prove Theorem 1 have some similarities, but also substantial differences from those used to prove the vertex cover nature of problem $1|\mathrm{prec}|\sum w_j C_j$ [5]. The differences come from the diversity of the two weighting functions that make, for example, the scheduling problem without precedence constraints a trivial problem and the (unconstrained) MINFAS with triangle inequality NP-complete. However, we believe that they both belong to a more general framework, that still has to be understood, and that may reveal the vertex cover nature of several other natural MINFAS problems (see Sect. 3 for a conjecture).

In the next section we prove Theorem 1 by showing how to "repair" in polynomial time any feasible solution to (3a)–(3d) to obtain a feasible solution to (1a)–(1e) that satisfies the claim. We conclude the paper with a conjecture locating the addressed problem into a general hierarchy within MINFAS.

2 Proof of Theorem 1

The structure of the proof is as follows. Consider any *minimal* integral solution[1] $\delta^* = \{\delta^*_{(i,j)} : \text{for all } i, j\}$ that is feasible to (3a)–(3d), but violates Constraint (1b). Let us say that pair $\{i, j\}$ is *contradicting* if $\delta^*_{(i,j)} = \delta^*_{(j,i)} = 1$. The violation of Constraint (1b) implies that there exists a non-empty set A of contradicting pairs. The minimality of δ^* implies that the removal of one of the two arcs of a contradicting pair yields an infeasible solution to (3a)–(3d). The proof works by identifying a subset $A' \subseteq A$ of contradicting pairs, together with another set B of arcs such that, by removing one of the two arcs in any pair from A' and by reverting the arcs in B, we obtain a feasible solution to (3a)–(3d) with a strictly smaller set of contradicting pairs. Moreover, the new solution is shown to be at least as good as the old one (here we use the assumption that the weighting function is hemimetric). By reiterating the same arguments we end up with a solution where no contradicting pair exists, i.e. feasible for (1a)–(1e), of value not worse than the initial one.

We start with a preliminary simple observation that characterizes minimal solutions and that will be used several times.

Lemma 1 *For any feasible minimal solution $\delta^* = \{\delta^*_{(i,j)} : \text{for all } i, j\}$ to (3a)–(3d) and any $i, j, k, \ell \in V$ such that $j \neq k$ and $i \neq \ell$, if $\delta^*_{(j,k)} = 1$, $\delta^*_{(k,j)} = 0$ and $(i, j), (k, \ell) \in P$ then $\delta^*_{(i,\ell)} = 1$ and $\delta^*_{(\ell,i)} = 0$.*

Proof Note that $\delta_{(i,\ell)} + \delta_{(k,j)} \geq 1$ is part of constraints (3b), therefore by the assumptions we have $\delta^*_{(i,\ell)} = 1$.

By contradiction, assume that $\delta^*_{(\ell,i)} = 1$. By minimality of solution δ^*, there must be a constraint that would be violated if we set $\delta^*_{(\ell,i)}$ to zero. The latter means that there are incomparable pairs (x_2, y_2) and (x_3, y_3) such that either (i) the following is a valid constraint (3b) with $\delta^*_{(x_2,y_2)} = 0$

$$\delta_{(\ell,i)} + \delta_{(x_2,y_2)} \geq 1,$$

[1] Recall that a $0 \setminus 1$ solution δ^* is *minimal* if the removal of any arc (i, j) from its support makes it unfeasible.

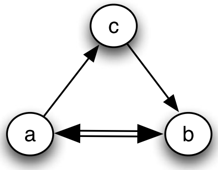


Fig. 1 Basic triple: $\delta^*_{(a,c)} = \delta^*_{(c,b)} = \delta^*_{(a,b)} = \delta^*_{(b,a)} = 1$, $\delta^*_{(c,a)} = \delta^*_{(b,c)} = 0$

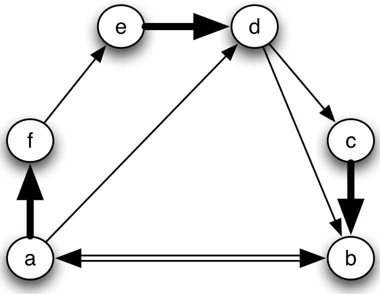


Fig. 2 Existence of a basic triple (a, d, b) assuming $\delta^*_{(a,b)} = \delta^*_{(b,a)} = 1$. *Bold arrows* represent poset relationship, namely $(a, f), (e, d), (c, b) \in P$

or (ii) the following is a valid constraint (3c)

$$\delta_{(\ell,i)} + \delta_{(x_2,y_2)} + \delta_{(x_3,y_3)} \geq 1,$$

with $\delta^*_{(x_2,y_2)} = \delta^*_{(x_3,y_3)} = 0$.

In Case (i), $(\ell, y_2), (x_2, i) \in P$ and, by transitivity of P, also $(k, y_2), (x_2, j) \in P$. It follows that Case (i) implies that $\delta_{(k,j)} + \delta_{(x_2,y_2)} \geq 1$ is a valid constraint that is violated by solution δ^*. Similarly, for case (ii), $(x_2, i), (x_3, y_2), (\ell, y_3) \in P$, and therefore also $(x_2, j), (x_3, y_2), (k, y_3) \in P$. So, Case (ii) implies that $\delta_{(k,j)} + \delta_{(x_2,y_2)} + \delta_{(x_3,y_3)} \geq 1$ is a valid constraint that is violated by solution δ^*. $\square$

Let $\delta^* = \{\delta^*_{(i,j)} : \text{for all } i, j\}$ be an α-approximate minimal solution to (3a)–(3d). For any triple $(a, c, b) \in V^3$ of distinct vertices, we say that (a, c, b) is a *basic triple* if the following holds (see Fig. 1): $\delta^*_{(a,c)} = \delta^*_{(c,b)} = \delta^*_{(a,b)} = \delta^*_{(b,a)} = 1$ and $\delta^*_{(c,a)} = \delta^*_{(b,c)} = 0$. Let T be the set of all the basic triples. The following lemma states that basic triples are "witnesses" of infeasibility.

Lemma 2 *If solution δ^* is a minimal solution to* (3a)–(3d) *but not feasible to* (1a)–(1e), *then $T \neq \emptyset$.*

Proof Assume that $\delta^*_{(a,b)} = \delta^*_{(b,a)} = 1$. Variable $\delta^*_{(a,b)}$ cannot be turned to zero because there exists $c, d, e, f \in V$ such that $\delta^*_{(c,d)} = \delta^*_{(e,f)} = 0$ and the following is a valid Constraint (3c) (this constraint is blocking $\delta^*_{(a,b)}$)

$$\delta_{(a,b)} + \delta_{(c,d)} + \delta_{(e,f)} \geq 1.$$

By a simple application of Lemma 1 (see Fig. 2) it follows that (a, b, d) is a basic triple.

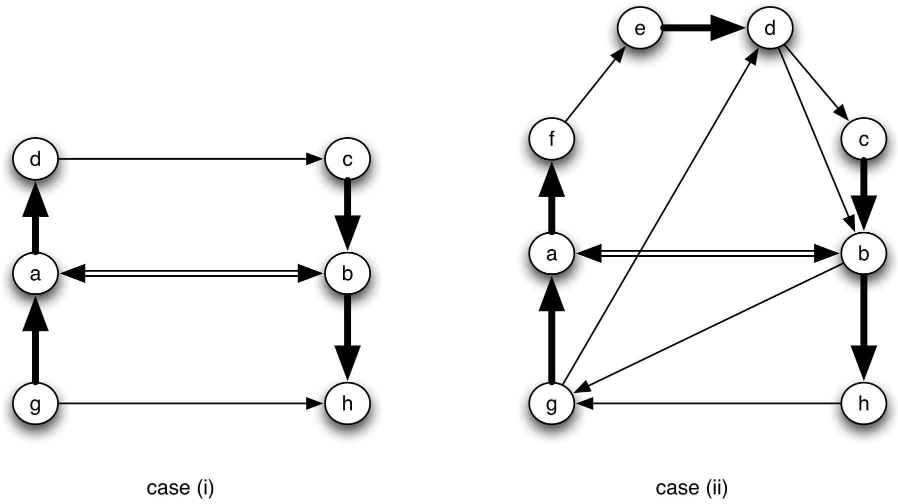


Fig. 3 Constraint (3b) cannot be a "blocking" constraint for a contradicting pair $\{a,b\}$ in any feasible solution. Indeed, the two cases in the figure show that this would imply a violated constraint: Bold arrows represent poset relationship; Case (i): $\delta_{h,g}=\delta_{d,c}=1$ and $\delta_{g,h}=\delta_{c,d}=0$; Case (ii): $\delta_{h,g}=\delta_{f,e}=\delta_{d,c}=1$ and $\delta_{g,h}=\delta_{e,f}=\delta_{c,d}=0$

Note that Constraint (3b) cannot be a blocking constraint for a contradicting pair. Indeed, consider $\{a,b\}$ with $\delta^*_{(a,b)}=\delta^*_{(b,a)}=1$. There are two cases (other cases are symmetric): Case (i) $\delta^*_{(a,b)}$ and $\delta^*_{(b,a)}$ are blocked by two constraints of type (3b); Case (ii) $\delta^*_{(a,b)}$ is blocked by Constraint (3c) and $\delta^*_{(b,a)}$ is blocked by Constraint (3b). These two cases are depicted in Fig. 3: note that Case (i) implies that constraint $\delta_{g,h}+\delta_{c,d}\geq 1$ is not satisfied; Viceversa, in Case (ii), by Lemma 1, we have $\delta_{g,d}=\delta_{d,b}=\delta_{b,g}=1$ and $\delta_{d,g}=\delta_{b,d}=\delta_{g,b}=0$, which implies that $\delta_{d,g}+\delta_{g,b}+\delta_{b,d}\geq 1$ is not satisfied. □

For any given vertex v, let us define the following set of arcs that will be used to "drop and reverse" arcs in a synchronized way to obtain new solutions:

$$S_v=\big\{(i,j):(v,i,j)\in T\big\}. \tag{4}$$

$$M_v=\big\{(i,j):(j,v,i)\in T\big\}. \tag{5}$$

$$E_v=\big\{(i,j):(i,j,v)\in T\big\}. \tag{6}$$

Note that S_v, M_v and E_v are pairwise disjoint.

Lemma 3 *For any $v\in V$ and $X\in\{S_v,E_v\}$, solution $\delta^X=\{\delta^X_{(i,j)}$: for all $i,j\}$ as defined in the following is a feasible solution for (3a)–(3d):*

1. $\delta^X_{(i,j)}=0$ *for each $(i,j)\in M_v$.*
2. $\delta^X_{(i,j)}=0$ *and $\delta^X_{(j,i)}=1$ for each $(i,j)\in X$.*
3. $\delta^X_{(i,j)}=\delta^*_{(i,j)}$ *elsewhere.*

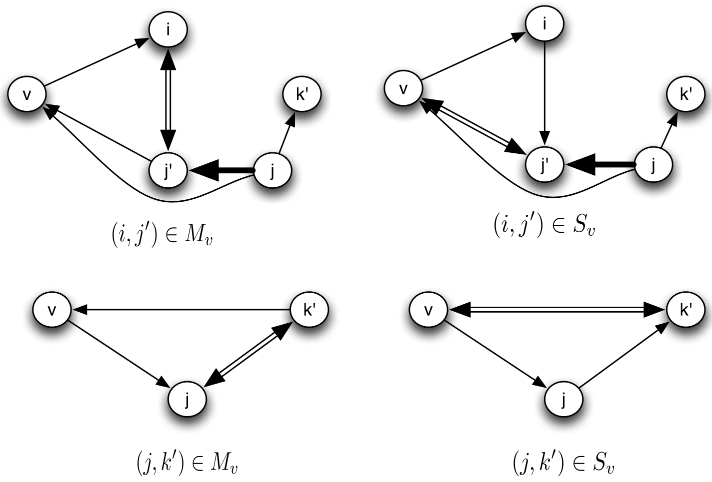


Fig. 4 Case (a)

Proof We start showing that solution δ^X satisfies the second set (3c) of constraints in (3a)–(3d) for any $X \in \{S_v, E_v\}$. The proof that δ^X satisfies the first set of constraints (3b) is similar.

Let us assume that $X = S_v$ (the proof for $X = E_v$ is symmetric). Since solution δ^X is obtained from the feasible solution δ^* by switching some variables to zero and others to one, we might violate only those constraints with at least one variable from δ^X that is turned to zero, i.e. the set of constraints that have at least one variable from $\{\delta^X_{(i,j)} : (i, j) \in X \cup M_v\}$. Let $(i, j') \in X \cup M_v$ and for any $j, k', k, i' \in V$ such that $\delta_{(i,j')} + \delta_{(j,k')} + \delta_{(k,i')} \geq 1$ is a valid constraint (3c), we want to prove that the following holds:

$$\delta^X_{(i,j')} + \delta^X_{(j,k')} + \delta^X_{(k,i')} \geq 1. \tag{7}$$

We distinguish between the following cases:

Case (a): $\delta^*_{(j,k')} = 1$ (see Fig. 4). Since $(i, j') \in S_v \cup M_v$ then $\delta^*_{(j',v)} = 1$. If $(i, j') \in M_v$ then $\delta^*_{(j',v)} = 1$ and $\delta^*_{(v,j')} = 0$. By applying Lemma 1 we can conclude that $\delta^*_{(j,v)} = 1$.
If $(i, j') \in S_v$ we claim that $\delta^*_{(j,v)} = 1$ as well. By contradiction assume $\delta^*_{(j,v)} = 0$ and therefore $\delta^*_{(v,j)} = 1$. By applying Lemma 1 we would have $\delta^*_{(v,j')} = 1$ and $\delta^*_{(j',v)} = 0$. The latter contradicts the assumption that $(i, j') \in S_v$.
Since $\delta^*_{(j,v)} = 1$, we have $(j, k') \notin S_v \cup M_v$ (since if $(j, k') \in S_v \cup M_v$ then $\delta^*_{(j,v)} = 0$) and therefore $\delta^X_{(j,k')} = \delta^*_{(j,k')} = 1$.

Case (b): $\delta^*_{(k,i')} = 1$ (see Fig. 5). Since $(i, j') \in S_v \cup M_v$ then $\delta^*_{(i,v)} = 0$ and $\delta^*_{(i',v)} = 0$ by Lemma 1. The latter implies that $(k, i') \notin S_v \cup M_v$ (since if $(k, i') \in S_v \cup M_v$ then $\delta^*_{(i',v)} = 1$) and therefore $\delta^X_{(k,i')} = \delta^*_{(k,i')} = 1$.

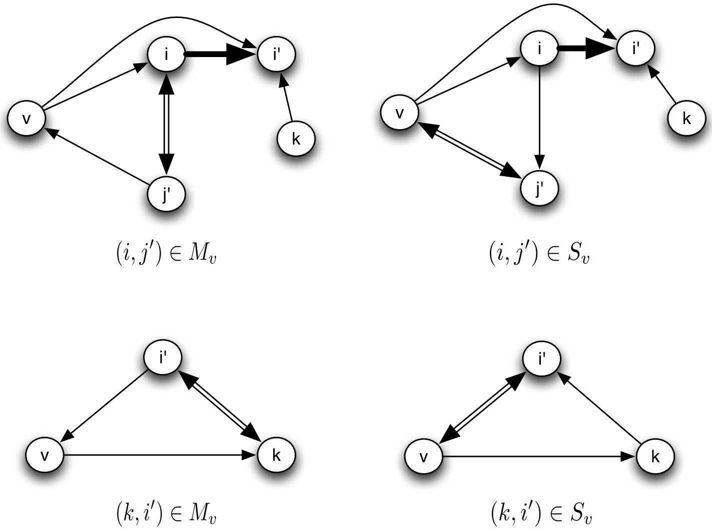


Fig. 5 Case (b)

Case (c): $\delta^*_{(j,k')} = \delta^*_{(k,i')} = 0$ (see Fig. 6). Under the current assumption, by Lemma 1 and constraint (3c), it is easy to check that $\delta^*_{(v,k)} = 1$. We distinguish between two subcases: (i) $\delta^*_{(k,v)} = 1$ and (ii) $\delta^*_{(k,v)} = 0$. If (i) holds[2] then $(i',k) \in S_v$ and therefore $\delta^X_{(k,i')} = 1$. Otherwise, by Lemma 1 we have $\delta^*_{(v,k')} = 1$ and $\delta^*_{(k',v)} = 0$. Moreover, since under (ii) we have $\delta^*_{(v,j')} = \delta^*_{(j',v)} = 1$, by minimality of the solution, the proof of Lemma 2 shows that it must exist a node q such that (v, j', q) is a basic triple, i.e. such that $\delta^*_{(j',q)} = \delta^*_{(q,v)} = \delta^*_{(v,j')} = \delta^*_{(j',v)} = 1$ and $\delta^*_{(q,j')} = \delta^*_{(v,q)} = 0$. By applying Lemma 1 we have $\delta^*_{(j,q)} = 1$ and $\delta^*_{(q,j)} = 0$. Therefore, $\delta^*_{(v,k')} = \delta^*_{(k',j)} = \delta^*_{(j,q)} = \delta^*_{(q,v)} = 1$ and $\delta^*_{(k',v)} = \delta^*_{(v,q)} = \delta^*_{(q,j)} = \delta^*_{(j,k')} = 0$ imply that $(k', j) \in S_v$ which implies that $\delta^X_{(j,k')} = 1$. □

According to solution δ^*, let us say that pair $\{i, j\}$ is *contradicting* if $\delta^*_{(i,j)} = \delta^*_{(j,i)} = 1$. By Lemma 3, any solution $\delta' \in \Lambda = \{\delta^X : v \in V \text{ and } X \in \{S_v, E_v\}\}$ is a feasible solution for (3a–(3d)). Moreover, it is easy to observe that δ' has a strictly smaller number of contradicting pairs.

The claim of Theorem 1 follows by proving the following Lemma 4 which shows that among the feasible solutions in Λ there exists one whose value is not worse than the value of δ^*. Therefore, after at most $O(|V|^2)$ "applications" of Lemma 4 we end up with a solution where no contradicting pair exists, i.e. feasible for (1a)–(1e).

[2] When $(i, j') \in M_v$ this is the only possible case, namely case (ii) does not hold. Indeed, see Fig. 6, when $(i, j') \in M_v$, the following is a valid constraint $\delta_{v,j'} + \delta_{j,k'} + \delta_{k,v} \geq 1$ and we are considering the case with $\delta_{v,j'} = \delta_{j,k'} = 0$.

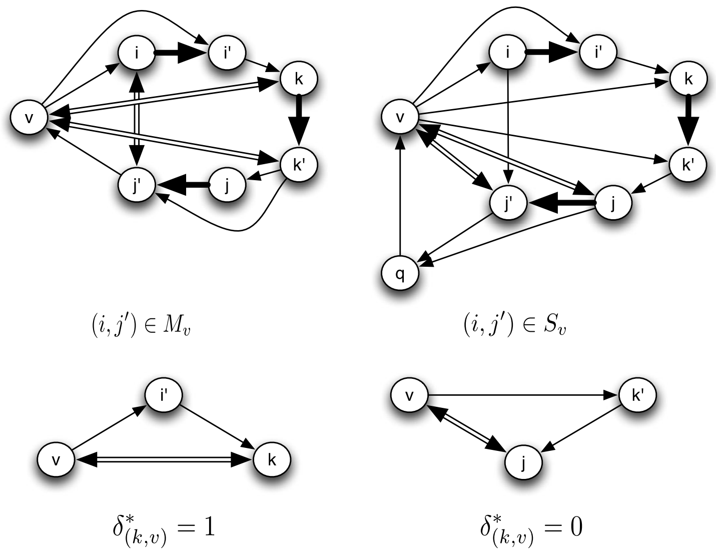


Fig. 6 Case (c)

Lemma 4 *If δ^* is not a feasible solution for* (1a)–(1e) *then there exists a feasible solution for* (3a)–(3d) *in $\Lambda = \{\delta^X : v \in V$ and $X \in \{S_v, E_v\}\}$ whose value is not worse than the value of δ^*.*

Proof By contradiction, we assume that every solution in Λ has value worse than δ^*.

By Lemma 3, for any vertex v we can obtain two feasible solutions by removing all the arcs from M_v and reverting, alternatively, either all the arcs from S_v, or all the arcs from E_v. Since we are assuming that every solution in Λ has value worse than δ^*, the following two inequalities express the latter for any $v \in V$.

$$\sum_{(b,a)\in M_v} w_{(b,a)} + \sum_{(i,j)\in S_v} w_{(i,j)} < \sum_{(i,j)\in S_v} w_{(j,i)}, \tag{8}$$

$$\sum_{(b,a)\in M_v} w_{(b,a)} + \sum_{(i,j)\in E_v} w_{(i,j)} < \sum_{(i,j)\in E_v} w_{(j,i)}. \tag{9}$$

By summing (8) and (9) for all v we obtain the following valid inequality:

$$\underbrace{\sum_{v\in V}\left(2\cdot \sum_{(b,a)\in M_v} w_{(b,a)} + \sum_{(i,j)\in S_v\cup E_v} w_{(i,j)}\right)}_{LHS(1)} < \underbrace{\sum_{v\in V}\left(\sum_{(i,j)\in S_v\cup E_v} w_{(j,i)}\right)}_{RHS(1)}. \tag{10}$$

A Triangle Inequality Condition For any basic triple $(a, c, b) \in T$ we consider the following two valid triangle inequalities.

$$w_{(c,a)} \le w_{(c,b)} + w_{(b,a)}, \tag{11}$$

$$w_{(b,c)} \leq w_{(b,a)} + w_{(a,c)}. \tag{12}$$

By summing (11) and (12) for all $(a,c,b) \in T$ we obtain the following valid inequality:

$$\underbrace{\sum_{(a,c,b)\in T} (w_{(b,c)} + w_{(c,a)})}_{LHS(2)} \leq \underbrace{\sum_{(a,c,b)\in T} (2 \cdot w_{(b,a)} + w_{(a,c)} + w_{(c,b)})}_{RHS(2)}. \tag{13}$$

The Contradiction Note that for every $(a,c,b) \in T$ we have $(a,c) \in E_b$ and $(c,b) \in S_a$. Therefore:

$$\begin{aligned} LHS(2) &= \sum_{(a,c,b)\in T} (w_{(b,c)} + w_{(c,a)}) \\ &= \sum_{v\in V}\Biggl(\sum_{(i,j):(v,i,j)\in T} w_{(j,i)} + \sum_{(i,j):(i,j,v)\in T} w_{(j,i)}\Biggr) \\ &\overset{(4),(6)}{=} \sum_{v\in V}\Biggl(\sum_{(i,j)\in S_v\cup E_v} w_{(j,i)}\Biggr) = RHS(1). \end{aligned} \tag{14}$$

Therefore, by (10), (13) and (14) we have $LHS(1) < RHS(1) = LHS(2) \leq RHS(2)$. We get a contradiction by showing that $RHS(2) = LHS(1)$:

$$\begin{aligned} &RHS(2) \\ &\quad= \sum_{(a,c,b)\in T} (2 \cdot w_{(b,a)} + w_{(a,c)} + w_{(c,b)}) \\ &\quad= \sum_{v\in V}\Biggl(2 \cdot \sum_{(a,b):(a,v,b)\in T} w_{(b,a)} + \sum_{(i,j):(v,i,j)\in T} w_{(i,j)} + \sum_{(i,j):(i,j,v)\in T} w_{(i,j)}\Biggr) \\ &\quad\overset{(4),(6),(5)}{=} \sum_{v\in V}\Biggl(2 \cdot \sum_{(b,a)\in M_v} w_{(b,a)} + \sum_{(i,j)\in S_v\cup E_v} w_{(i,j)}\Biggr) = LHS(1). \end{aligned} \tag{15}$$

□

3 Future Directions

The constrained MINFAS problem admits a natural covering formulation with an exponential number of constraints (see e.g. [6]):

$$\min \quad \sum_{(i,j)} \delta_{(i,j)} w_{(i,j)} \tag{16a}$$

$$\text{s.t.} \quad \sum_{i=1}^{c} \delta_{(x_i,y_i)} \geq 1, \quad \text{for all } c \geq 2,\ (x_i, y_i)_{i=1}^{c} \text{ s.t. } (x_i, y_{i+1}) \in P, \tag{16b}$$

$$\delta_{(i,j)} \in \{0, 1\}, \qquad (i, j) \in \text{inc}(\mathbf{P}). \tag{16c}$$

The condition $(x_i, y_{i+1}) \in P$ in constraint (16b) is to be read cyclically, namely, $(x_c, y_1) \in P$. The hyperedges in this vertex cover problem are exactly the alternating cycles of poset P (see e.g. [26]).

In this paper we prove that when the weights satisfy the triangle inequality then we can drop from (16a)–(16c) all the constraints of size strictly larger than three. Generalizing, it would be nice to prove/disprove the following statement that we conjecture to be true.

Hypothesis 1 *If the weights satisfy the k-gonal inequalities, i.e., for all $a_1, \ldots, a_k \in V$ we have $w_{(a_1,a_k)} \leq w_{(a_1,a_2)} + \ldots + w_{(a_{k-1},a_k)}$, then there exists a constant $c(k)$, whose value depends on k, such that a proper formulation for the constrained* MinFAS *problem can be obtained by dropping from* (16a)–(16c) *all the constraints of size strictly larger than $c(k)$.*

MinFAS problems with weights belonging to interval $[1, k-1]$ are examples of problems with k-gonal inequalities on the weights. If true, the above structural result has the important implication that, for any constant k, constrained MinFAS with k-gonal inequalities on the weights admits a constant approximation algorithm (in contrast to the general case with arbitrary k that does not seem to have any constant approximation assuming the Unique Games Conjecture [14]).

Acknowledgements To my beloved mom Elsa.

Appendix A: Ranking with Probability Inequalities: a Counterexample

The following example shows that probabilities inequalities are not sufficient for (3a)–(3d) to be a proper formulation:

$$w_{(i,j)} + w_{(j,i)} = 1 \quad \text{for all distinct } i, j$$

Consider the instance with 8 nodes with weight zero on the arcs displayed in Fig. 7 (therefore the reversed arcs have weight 1). Moreover, all the arcs in $\{2, 3\} \times \{7, 8\}$ have weight 1 (the reversed zero). Finally, all the remaining arcs have weight 0.5, namely those in $\{1\} \times \{4, 5, 6\}$ and the reversed ones. A feasible solution for (2a)–(2d) is obtained by picking all the displayed arcs in Fig. 7 and none of the reversed ones (therefore we have to pick also those in $\{2, 3\} \times \{7, 8\}$, $\{7, 8\} \times \{2, 3\}$, $\{4, 5, 6\} \times \{1\}$ and $\{1\} \times \{4, 5, 6\}$ in order to satisfy the constraints in (2a)–(2d)). This solution has value 7, whereas any total ordering has value not smaller than 7.5 (the best total ordering is $(2, 3, 4, 5, 6, 7, 8, 1)$).

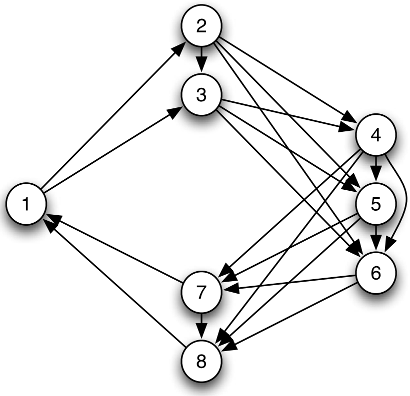
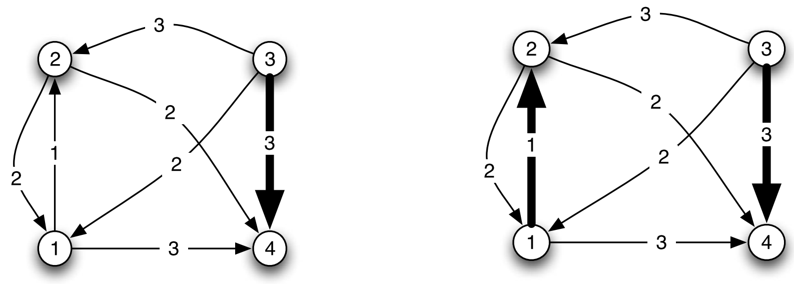


Fig. 7 Counterexample for probability inequalities



(a) The non displayed arcs have weight = 0. Arc (3,4) is a precedence constraint.

(b) The non displayed arcs have weight = 0. Arcs (3,4) and (1,2) are precedence constraints.

Fig. 8 Solution $\delta^*_{(1,2)} = \delta^*_{(2,3)} = \delta^*_{(3,4)} = \delta^*_{(4,1)} = \delta^*_{(1,3)} = \delta^*_{(3,1)} = \delta^*_{(2,4)} = \delta^*_{(4,2)} = 1$ has value smaller than any valid permutation

Appendix B: A Comment on Formulation (3a)–(3d)

If the poset is not empty the additional constraints that are present in formulation (3a)–(3d) but not in (2a)–(2d) are also necessary. Indeed, in Figure 8 any permutation that complies with the precedence constraints has value larger than the solution suggested in the picture with a cycle.

References

1. Aharoni, R., Holzman, R., Krivelevich, M.: On a theorem of Lovász on covers in τ-partite hypergraphs. Combinatorica **16**(2), 149–174 (1996)
2. Ailon, N.: Aggregation of partial rankings, p-ratings and top-m lists. Algorithmica **57**(2), 284–300 (2010)
3. Ailon, N., Avigdor-Elgrabli, N., Liberty, E., van Zuylen, A.: Improved approximation algorithms for bipartite correlation clustering. SIAM J. Comput. **41**(5), 1110–1121 (2012)

4. Ailon, N., Charikar, M., Newman, A.: Aggregating inconsistent information: Ranking and clustering. J. ACM **55**(5) (2008)
5. Ambühl, C., Mastrolilli, M.: Single machine precedence constrained scheduling is a vertex cover problem. Algorithmica **53**(4), 488–503 (2009)
6. Ambühl, C., Mastrolilli, M., Mutsanas, N., Svensson, O.: Precedence constraint scheduling and connections to dimension theory of partial orders. Bull. Eur. Assoc. Theor. Comput. Sci. **95**, 45–58 (2008)
7. Ambühl, C., Mastrolilli, M., Mutsanas, N., Svensson, O.: On the approximability of single-machine scheduling with precedence constraints. Math. Oper. Res. **36**(4), 653–669 (2011)
8. Ambühl, C., Mastrolilli, M., Svensson, O.: Inapproximability results for sparsest cut, optimal linear arrangement, and precedence constraint scheduling. In: Proceedings of FOCS, pp. 329–337 (2007)
9. Bansal, N., Khot, S.: Optimal Long-Code test with one free bit. In: Proceedings of FOCS, pp. 453–462 (2009)
10. Correa, J.R., Schulz, A.S.: Single machine scheduling with precedence constraints. Math. Oper. Res. **30**(4), 1005–1021 (2005)
11. Even, G., Naor, J., Rao, S., Schieber, B.: Divide-and-conquer approximation algorithms via spreading metrics. J. ACM **47**(4), 585–616 (2000)
12. Even, G., Naor, J., Schieber, B., Sudan, M.: Approximating minimum feedback sets and multicuts in directed graphs. Algorithmica **20**(2), 151–174 (1998)
13. Grötschel, M., Jünger, M., Reinelt, G.: Acyclic subdigraphs and linear orderings: polytopes, facets, and a cutting plane algorithm. In: Graphs and Orders, pp. 217–264 (1985)
14. Guruswami, V., Håstad, J., Manokaran, R., Raghavendra, P., Charikar, M.: Beating the random ordering is hard: every ordering csp is approximation resistant. SIAM J. Comput. **40**(3), 878–914 (2011)
15. Hall, L.A., Schulz, A.S., Shmoys, D.B., Wein, J.: Scheduling to minimize average completion time: off-line and on-line algorithms. Math. Oper. Res. **22**, 513–544 (1997)
16. Kann, V.: On the Approximability of NP-Complete Optimization Problems. Ph.D. thesis, Department of Numerical Analysis and Computing Science, Royal Institute of Technology, Stockholm (1992)
17. Karp, R.: Reducibility Among Combinatorial Problems, pp. 85–103. Plenum, New York (1972)
18. Kenyon-Mathieu, C., Schudy, W.: How to rank with few errors. In: Proceedings of STOC, pp. 95–103 (2007)
19. Krivelevich, M.: Approximate set covering in uniform hypergraphs. J. Algorithms **25**(1), 118–143 (1997)
20. Kuhn, F., Mastrolilli, M.: Vertex cover in graphs with locally few colors. Inf. Comput. **222**, 265–277 (2013)
21. Lawler, E.L., Lenstra, J.K., Rinnooy Kan, A.H.G., Shmoys, D.B.: Sequencing and scheduling: Algorithms and complexity. In: Graves, S.C., Rinnooy Kan, A.H.G., Zipkin, P. (eds.) Handbooks in Operations Research and Management Science, vol. 4, pp. 445–552. North-Holland, Amsterdam (1993)
22. Lempel, A., Cederbaum, I.: Minimum feedback arc and vertex sets of a directed graph. IEEE Trans. Circuit Theory **4**(13), 399–403 (1966)
23. Pardalos, P., Du, D.: Handbook of Combinatorial Optimization: Supplement vol. 1. Springer, Berlin (1999)
24. Schuurman, P., Woeginger, G.J.: Polynomial time approximation algorithms for machine scheduling: ten open problems. J. Sched. **2**(5), 203–213 (1999)
25. Seymour, P.D.: Packing directed circuits fractionally. Combinatorica **15**(2), 281–288 (1995)
26. Trotter, W.T.: Combinatorics and Partially Ordered Sets: Dimension Theory. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore (1992)
27. van Zuylen, A., Hegde, R., Jain, K., Williamson, D.P.: Deterministic pivoting algorithms for constrained ranking and clustering problems. In: Proceedings of SODA, pp. 405–414 (2007)
28. van Zuylen, A., Williamson, D.P.: Deterministic pivoting algorithms for constrained ranking and clustering problems. Math. Oper. Res. **34**(3), 594–620 (2009)
29. Vazirani, V.V.: Approximation Algorithms. Springer, Berlin (2001)